When is a family of generalized means a scale?

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Abstract

For a family $\{k_t \mid t \in I\}$ of real $\mathcal{C}^2$ functions defined on U (I, U – open intervals) and satisfying some mild regularity conditions, we prove that the mapping $I \ni t \mapsto k_t^{-1}\big(\sum_{i=1}^n w_i k_t(a_i)\big)$ is a continuous bijection between I and $(\min \underline{a}, \max \underline{a})$, for every fixed non-constant sequence $\underline{a} = \big(a_i\big)_{i=1}^n$ with values in U and every set, of the same cardinality, of positive weights $\underline{w} = \big(w_i\big)_{i=1}^n$. In such a situation one says that the family of functions $\{k_t\}$ generates a *scale* on U. The precise assumptions in our result read (all indicated derivatives are with respect to $x \in U$)

(i) k_t' does not vanish anywhere in U for every $t \in I$,

(ii) $I \ni t \mapsto \frac{k_t''(x)}{k_t'(x)}$ is increasing, 1–1 on a dense subset of U and onto the image $\mathbb{R}$ for every $x \in U$.

This result makes possible three things. 1) a new and extremely short proof of the classical fact that *power means* generate a scale on $(0, +\infty)$, 2) a short proof of a fact, which is in a direct relation to two results established by Kolesárová in 2001, that, for every strictly increasing convex and $\mathcal{C}^2$ function $g\colon (0,\,1) \to (0,\,+\infty)$, the class $\{\mathfrak{M}_{g_\alpha}\}_{\alpha \in (0,\,+\infty)}$ of quasi-arithmetic means (see Introduction for the definition) generated by functions g_α, $g_\alpha(x) = g(x^\alpha)$, $\alpha \in (0, +\infty)$, generates a scale on $(0, 1)$ between the geometric mean and maximum (meaning that, for every $\underline{a}$, $\underline{w}$, if $s \in \big(\prod_{i=1}^n a_i^{w_i},\, \max(\underline{a})\big)$ then there exists exactly one α such that $\mathfrak{M}_{g_\alpha}(\underline{a}, \underline{w}) = s$).

3) a brief proof of one of the classical results of the Italian statistics' school from the 1910-20s that the so-called *radical means* generate a scale on $(0,\,+\infty)$.

1 Introduction

One of the most popular families of means encountered in the literature consists of quasi-arithmetic means. That mean is defined for any continuous strictly monotone function $f\colon U \to \mathbb{R}$, U – an open interval. When $\underline{a} = (a_1, \ldots, a_n)$ is a sequence of points in U and $\underline{w} = (w_1, \ldots, w_n)$ is a sequence of *weights* ($w_i > 0$, $w_1 + \cdots + w_n = 1$), then the mean $\mathfrak{M} = \mathfrak{M}_f(\underline{a},\, \underline{w})$ is well-defined by the equality

$$f(\mathfrak{M}) = \sum_{i=1}^n w_i f(a_i)\,.$$

According to [12, pp. 158–159], this family of means was dealt with for the first time in the papers [10, 13, 14] in the early thirties of the last century as a natural generalization of the power means. Clearly, it is also discussed in the by-now-classical encyclopaedic publications [B, 6]. One gets this family, containing the most popular means: arithmetic, geometric, quadratic, harmonic, by putting

$$f_r(x) = \begin{cases} x^r & \text{if } r \neq 0 \\ \ln x & \text{if } r = 0 \end{cases},$$

$x \in U = (0, +\infty)$, $r \in I = \mathbb{R}$.

We pass now to the notion of scale in the theory of means. If a non-constant vector $\underline{a} \in U^n$ and weights $\underline{w}$ are fixed then the mapping $f \mapsto \mathfrak{M}_f(\underline{a}, \underline{w})$ takes continuous monotone functions $f\colon U \to \mathbb{R}$ to the interval $(\min \underline{a}, \max \underline{a})$. One is interested in finding such families of functions $\{f_i\colon U \to \mathbb{R}\}_{i\in I}$, where I is an interval, that for every non-constant vector $\underline{a}$ with values in U and arbitrary fixed corresponding weights $\underline{w}$, the mapping $I \ni i \mapsto \mathfrak{M}_{f_i}(\underline{a}, \underline{w})$ be a *bijection* onto $(\min \underline{a}, \max \underline{a})$. Every such a family of means $\mathfrak{M}_{f_i}$ is called *scale on* U.

The problem of finding conditions, for a family of means, equivalent to its being a scale has been discussed for various families. For instance, a set of conditions pertinent for Gini means was presented in [1]. Many results concerning means may be expressed in a compact way in terms of scales. Probably the most famous is the fact that the family of power means is a scale on $(0, +\infty)$. It was proved for the first time (for arbitrary weights) in [2]. More about the underlying history, as well as another proof, was given in [B, p. 203]. In the last section of the present note we will present a new, extremely short proof of this classical fact.

2 Comparison of means

Dealing with means, we would like to know whether (a) one mean is not smaller than the other, whenever both are defined on the same interval and computed on same, but arbitrary, set of arguments. And, when (a) holds true, whether (b) the two means, evaluated on arguments, are equal only when all components in an input $\underline{a}$ are the same: $a_1 = a_2 = \cdots = a_n$. With (a) and (b) holding true, we would say that the first mean is *greater* than the second.

As long as quasi-arithmetic means are concerned, the comparability of $\mathfrak{M}_f$ and $\mathfrak{M}_g$ as such turns out to be intimately related to the convexity of the function $f \circ g^{-1}$, see items (ii) and (iii) in Proposition 1 below.

Unfortunately, however, when it comes to scales, the family of objects to handle becomes uncountable. Hence one is forced to use another tool, allowing to tell something about an uncountable family of means. Its concept goes back to a seminal paper [M]. A key operator A from [M] (recalled below) is used in item (i) in our technically crucial Proposition 1.

In fact, let U be an interval, $\mathcal{C}^{2\neq}(U)$ be the class of functions from $\mathcal{C}^2(U)$ with the first derivative vanishing nowhere in U. Within this class one defines $A\colon \mathcal{C}^{2\neq}(U) \to \mathcal{C}(U)$ by the formula

$$A(f) = \frac{f''}{f'}\,.$$

However, the operator A will be used so often as to adopt the convention that, for $a, b, c, \dots \in \mathcal{C}^{2\neq}(U)$, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots$ stand for $A(a), A(b), A(c), \dots$ Due to [M], this operator has wide applications in the comparison of means – see Proposition 1. In fact, it will enable us to compare means in huge families, not only in pairs. Precisely this kind of comparison was being advanced by Polish mathematicians in the late 1940s.

One of the most important facts was discovered by Mikusiński, who published his result, [M], in "Studia Mathematica"[1]. It is quite surprising that such a useful result has not been included in the referential book [B].

We present both necessary and sufficient conditions, for a family of functions $\{k_t\}_{t\in I}$ defined on a common interval U, to generate a scale on U. The key conditions in our Theorems 1 and 2 are given in terms of the operator A. Reiterating, it is handy to compare means with its help. We begin with

[1]the flagship journal of the pre-war Lvov Mathematical School, established by H. Steinhaus and S. Banach.

Theorem 1. *Let U be an interval, $I = (a, b)$ an open interval, $(k_\alpha)_{\alpha\in I}$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all α.*

If $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ is increasing and 1–1 on a dense subset of U, and is onto for all $x \in U$, then $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is an increasing scale on U.

A proof of this theorem is given in Section 4. As a matter of fact, we will need a wider version of the above theorem. Namely, we extend the setup as follows.

In the definition of a scale (see Introduction) one may replace $\min \underline{a}$ and $\max \underline{a}$ by arbitrary bounds $L(\underline{a}, \underline{w})$ and $H(\underline{a}, \underline{w})$ respectively, with some functions L and H.[2] Then such a modified family of means is called a *scale between L and H*. Such generalization is very natural and is frequently used, e. g. in [B, pp. 323, 364].

Bounds in a scale, in most cases, are either quasi-arithmetic means or min, or max. In order to make the notation more homogeneous, we introduce two extra symbols $\perp$ and $\top$, and write henceforth, purely formally, $\mathfrak{M}_\perp = \min$ and $\mathfrak{M}_\top = \max$. We also adopt the convention that $A(\perp) = -\infty$ and $A(\top) = +\infty$.

Attention. In some papers scales may as well be decreasing. In fact, we do not lose generality if we assume that all scales are increasing, because whenever a family $\{k_\alpha\}_{\alpha\in I}$ generates a decreasing scale and $\varphi\colon J \to I$ is continuous, decreasing, 1–1 and onto, then the family $\{k_{\varphi(\alpha)}\}_{\alpha\in J}$ generates an increasing scale (see, e. g., Proposition 6 in Section 5).

Corollary 1 (Bounded Scale)**.** *Let $l, h \in \mathcal{C}^{2\neq}(U) \cup \{\perp, \top\}$. Let U and $I = (a, b)$ be open intervals, $(k_\alpha)_{\alpha\in I}$ be a family of functions, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all α.*

If $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ is increasing (decreasing), 1–1 on a dense subset of U and onto $(A(l)(x), A(h)(x))$ for all $x \in U$, then $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is an increasing (decreasing) scale between $\mathfrak{M}_l$ and $\mathfrak{M}_h$.

The proof is but a specification of the proof of Theorem 1.

Remark. If, in the above corollary, $l, h \in \mathcal{C}^{2\neq}(U)$, then it is enough to assume that the mapping $\alpha \mapsto A(k_\alpha)(x)$ be onto for almost all $x \in U$. (Then, by Theorem 3, one gets the convergence in L_1).

The strength of Theorem 1 is visible in the following exercise.

Example 1. *Let $U = (\frac{1}{e}, +\infty)$ and $k_\alpha(x) = x^{\alpha x}$ for $\alpha \in \mathbb{R} \setminus \{0\}$.*
Find a function k_0 such that the completed family $(k_\alpha)_{\alpha\in\mathbb{R}}$ generates a scale on U.

By the definition of the operator A, for $\alpha \neq 0$ there holds

$$\boldsymbol{k_\alpha}(x) = \frac{1}{x(\ln x + 1)} + \alpha(\ln x + 1)\,.$$

In view of Theorem 1 we will be done, provided $\alpha \mapsto \boldsymbol{k_\alpha}(x)$ is increasing, 1–1 and onto $\mathbb{R}$ for all $x \in U$. But

$$\mathbb{R} \setminus \{0\} \ni \alpha \mapsto \boldsymbol{k_\alpha}(x) \in \mathbb{R} \setminus \left\{\frac{1}{x(\ln x + 1)}\right\} \quad \text{for all } x \in U\,.$$

Hence it is natural to take $k_0 = A^{-1}\left(\frac{1}{x(\ln x+1)}\right)$. Then the pattern $A^{-1}(\boldsymbol{f}) = \int e^{\int \boldsymbol{f}}$ gives automatically $k_0(x) = x \ln x$.
Therefore, an increasing scale on $(\frac{1}{e}, +\infty)$ is generated by the family

$$k_\alpha = \begin{cases} x \mapsto x^{\alpha x} & \text{if } \alpha \neq 0\,, \\ x \mapsto x \ln x & \text{if } \alpha = 0\,. \end{cases}$$

[2] We slightly abuse the notation here, as most of the researchers active in the field of means do, e. g., in [B, p. 61]

Moreover, it is now immediate to note that, in turn, the same family of functions generates a *decreasing* scale on $(0, \frac{1}{e})$.

How about a possible reversing of Theorem 1 ? This point is rather fine; the existence of a scale implies a somehow weaker set of properties than the one assumed in Theorem 1. To the best of author's knowledge, the problem of finding a set of conditions *exactly* equivalent to generating a scale is still (and, most likely, widely) open.

Theorem 2. *Let U be an interval, $I = (a,\, b)$ an open interval, $(k_\alpha)_{\alpha\in I}$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all α.*

If $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is an increasing scale then there exists an open dense subset $X \subset U$ such that the mappping $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ is increasing, 1–1 and onto for all $x \in X$.

A proof of this theorem is given in Section 4, immediately after the proof of Theorem 1.

3 Properties and uses of A

In what follows we will extensively use the operator A. Here we recall, after [M], some of its key properties. We also rephrase in the terms of A an important result from [9].

All this will be instrumental in showing that many nontrivial families of functions do generate scales. We will also deduce about the limit properties of our quasi-arithmetic means, stating a new result (Proposition 5) inspired, to some extent, by the paper [K].

Regarding scales as such, many examples of them were furnished in [B, p. 269]. Scales were also used by the old Italian school of statisticians; see, e. g., [3, 4, 5, 11, 15, 16]. One of significant results from that last group of works will be presented, with a new and compact proof, in Proposition 6. That new approach will, we hope, show how quickly one can nowadays prove old results.

Remark 1. Let U be an interval and $f,\, g \in \mathcal{C}^{2\neq}(U)$. Then the following conditions are equivalent:

(i) $A(f)(x) = A(g)(x)$ for all $x \in U$,

(ii) $f = \alpha g + \beta$ for some $\alpha,\, \beta \in \mathbb{R},\ \alpha \neq 0$,

(iii) $\mathfrak{M}_f(\underline{a},\, \underline{w}) = \mathfrak{M}_g(\underline{a},\, \underline{w})$ for all vectors $\underline{a} \in U^n$ and arbitrary corresponding weights $\underline{w}$

(see, for instance, [12, p. 66], [M]).

Let f be a strictly monotone function such that $f \in \mathcal{C}^1(U)$ and $f'(x) \neq 0$ for all $x \in U$. Then there either holds $f'(x) < 0$ for all $x \in U$, or else $f'(x) > 0$ for all $x \in U$. So we define the sign $\mathrm{sgn}(f')$ of the first derivative of f to be $\mathrm{sgn}(f')(x)$, where x is any point in U. The key tool in our approach is

Proposition 1 (Basic comparison)**.** *Let U be an interval, $f,\, g \in \mathcal{C}^{2\neq}(U)$. Then the following conditions are equivalent:*

(i) *$A(f) > A(g)$ on a dense set in U ,*

(ii) *$(\mathrm{sgn} f') \cdot (f \circ g^{-1})$ is strictly convex ,*

(iii) *$\mathfrak{M}_f(\underline{a},\, \underline{w}) \geq \mathfrak{M}_g(\underline{a},\, \underline{w})$ for all vectors $\underline{a} \in U^n$ and weights $\underline{w}$, with both sides equal only when $\underline{a}$ is a constant vector.*

For the equivalence of (i) and (iii), see [M, p. 95] (this characterization of comparability of means had, in the same time, been obtained independently by S. Łojasiewicz – see footnote 2 in [M]). For the equivalence of (ii) and (iii), see, for instance, [8, p. 1053].

In the course of comparing means, one needs to majorate the difference between two means. If the interval U is unbounded then, of course, the difference between any given two means can be unbounded (for example such is the difference between the arithmetic and geometric mean). In order to eliminate this drawback, we will henceforth suppose that the means are always defined on a compact interval. It will be with no loss of generality, because it is easy to check that a family of means defined on U is a scale on U if and only if those means form a scale on D, when treated as functions $D \to \mathbb{R}$, for every closed subinterval $D \subset U$. Indeed, if $\underline{a}$ is a vector with values in U, then $\underline{a}$ is also a vector with values in D for some closed subinterval D of U.

So, from now on, we have U – a compact interval, $g \in \mathcal{C}^{2\neq}(U)$ increasing, and $\boldsymbol{g} \in L_1(U)$. The following theorem is of utmost technical importance.

Theorem 3. *Let U be a closed bounded interval. If, for $n \in \mathbb{N}$, $\boldsymbol{f} \colon U \to \mathbb{R}$, $\boldsymbol{k_n} \in \mathcal{C}(U)$ and $\boldsymbol{k_n} \underset{L_1}{\longrightarrow} \boldsymbol{f}$ then $\mathfrak{M}_{k_n} \rightrightarrows \mathfrak{M}_f$ uniformly with respect to $\underline{a}$ and $\underline{w}$. Moreover,*

$$|\mathfrak{M}_f(\underline{a},\underline{w}) - \mathfrak{M}_{k_n}(\underline{a},\underline{w})| \le |U|\, e^{2\|\boldsymbol{f}\|_1} \sinh 2\, \|\boldsymbol{k}_n - \boldsymbol{f}\|_1$$

for all $\underline{a}$ and $\underline{w}$ ($\|\cdot\|_1$ is taken in the space $L_1(U)$).

Proof. Let $u = \inf U$. Solving a simple differential equation, in view of Remark 1, it is possible to assume, for all considered functions f, that

$$f(x) = \int_u^x \exp\left(\int_u^s \boldsymbol{f}(t)dt\right) ds, \quad x \in U.$$

Much like in [9, p. 216], we have

$$\mathfrak{M}_f(\underline{a},\underline{w}) - \mathfrak{M}_{k_n}(\underline{a},\underline{w}) = (f^{-1})'(\alpha) \sum_{1\le i\le j\le m} p_i p_j \left(k_n(a_i) - k_n(a_j)\right)\left(\theta_n(z_i) - \theta_n(z_j)\right)$$

for certain $\alpha \in [\min \underline{a}, \max \underline{a}]$, $\theta_n = (f \circ k_n^{-1})'$, $p_i \in (0,1)$, $\sum_{1\le i\le j\le m} p_i p_j \le 1/4$. Now, continuing,

$$\begin{aligned}
&|\mathfrak{M}_f(\underline{a},\underline{w}) - \mathfrak{M}_{k_n}(\underline{a},\underline{w})| \\
&= \left|(f^{-1})'(\alpha) \sum_{1\le i\le j\le m} p_i p_j (k_n(a_i) - k_n(a_j))\left(\theta_n(z_i) - \theta_n(z_j)\right)\right| \\
&\le \left\|(f^{-1})'\right\|_\infty \tfrac{1}{4} \left(k_n(\max \underline{a}) - k_n(\min \underline{a})\right)\ 2 \sup_{z,v\in U} |\theta_n(z) - \theta_n(v)|
\end{aligned}$$

Putting $\varepsilon := \|\boldsymbol{k}_n - \boldsymbol{f}\|_1$, we assuredly have

$$\frac{k_n'}{f'} = e^{\int \boldsymbol{k}_n - \boldsymbol{f}} \in (e^{-\varepsilon},\, e^{\varepsilon})\,.$$

So $\theta_n = (f \circ k_n^{-1})'(x) = \frac{f' \circ k_n^{-1}(x)}{k_n' \circ k_n^{-1}(x)} \in (e^{-\varepsilon},\, e^{\varepsilon})$. What is more,

$$\begin{aligned}
k_n(\max \underline{a}) - k_n(\min \underline{a}) &= \int_{\min \underline{a}}^{\max \underline{a}} k_n'(x)\, dx \\
&\le \int_{\min \underline{a}}^{\max \underline{a}} e^{\varepsilon} f'(x)\, dx \\
&= e^{\varepsilon}(f(\max \underline{a}) - f(\min \underline{a})).
\end{aligned}$$

Hence, continuing further,

$$
\begin{aligned}
|\mathfrak{M}_f(\underline{a},\underline{w}) - \mathfrak{M}_{k_n}(\underline{a},\underline{w})| &\leq \frac{\left\|(f^{-1})'\right\|_\infty}{2}(k_n(\max\underline{a}) - k_n(\min\underline{a}))\sup_{z,v\in U}|\theta_n(z)-\theta_n(v)| \\
&\leq \frac{\left\|(f^{-1})'\right\|_\infty e^\varepsilon}{2}(f(\max\underline{a}) - f(\min\underline{a}))\left|e^\varepsilon - e^{-\varepsilon}\right| \\
&\leq \frac{\|f\|_\infty}{\inf f'} e^\varepsilon \sinh\varepsilon \\
&\leq \frac{\|f\|_\infty}{\inf f'}\sinh 2\varepsilon\,.
\end{aligned}
$$

But we also know that

$$\|f\|_\infty = \left\|e^{\int \boldsymbol{f}}\right\|_1 \leq |U|\, e^{\|\boldsymbol{f}\|_1} \tag{1}$$

and

$$\inf f' = \inf e^{\int \boldsymbol{f}} \geq e^{-\|\boldsymbol{f}\|_1}. \tag{2}$$

So, prolonging the previous chain of estimations and using (1) and (2),

$$|\mathfrak{M}_f(\underline{a},\underline{w}) - \mathfrak{M}_{k_n}(\underline{a},\underline{w})| \leq |U|\, e^{2\|\boldsymbol{f}\|_1}\sinh 2\,\|\boldsymbol{k}_n - \boldsymbol{f}\|_1\,.$$

Hence $\mathfrak{M}_{k_n} \rightrightarrows \mathfrak{M}_f$. Theorem is now proved. ☐

Heading towards the main results of the note, we state now

Proposition 2. *Let U be a closed bounded interval, $I = (a, b)$ – an open interval, $(k_\alpha)_{\alpha\in I}$ – a family of functions from $\mathcal{C}^{2\neq}(U)$.*
(A) *If $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is an increasing scale then $(A(k_\alpha))_{\alpha\in I}$ satisfies all the conditions* (a) *through* (d) *listed below.*

(a) *if $\alpha_i \to \alpha$, then $A(k_{\alpha_i}) \to A(k_\alpha)$,*

(b) *if $\alpha < \beta$, then $A(k_\alpha) < A(k_\beta)$ on a dense subset of U,*

(c) *if $\alpha \to a+$, then $A(k_\alpha)(x) \to -\infty$ on a dense subset of U,*

(d) *if $\beta \to b-$, then $A(k_\beta)(x) \to +\infty$ on a dense subset of U.*

(B) *Strengthening the pair of conditions* (c) *and* (d) *to*

(e) *($\alpha \to a+ \Rightarrow A(k_\alpha)(x) \to -\infty$) and ($\beta \to b- \Rightarrow A(k_\beta)(x) \to +\infty$) for all $x \in U$*

suffices to reverse the implication: (a), (b) *and* (e) *implies $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ being an increasing scale.*

Proof. To simplify the notation, having $\underline{a}$ and $\underline{w}$ fixed, we write shortly

$$F(\alpha) = \mathfrak{M}_{k_\alpha}(\underline{a},\, \underline{w}),$$

$F\colon I \to (\min\underline{a},\, \max\underline{a})$. And then one simply checks step by step:

(a) if $\alpha_i \to \alpha+$ we have that $F(\alpha_i) \to F(\alpha)$.

But it is easy to check that

$$\boldsymbol{f}(x) \;=\; \lim_{\varepsilon\to 0+}\frac{2}{\varepsilon^2}\mathfrak{M}_f(x-\varepsilon,\, x+\varepsilon)\,.$$

So $\boldsymbol{k_{\alpha_i}} \to \boldsymbol{k_\alpha}$.

(b) if $\alpha < \beta$, we have $F(\alpha) \leq F(\beta)$ and the equality holds iff $\underline{a}$ is constant. So by Proposition 1 we have $\boldsymbol{k_\alpha} < \boldsymbol{k_\beta}$ on a dense set.
Let $E_{\alpha,\beta} = \{x \in U : \boldsymbol{k_\alpha}(x) = \boldsymbol{k_\beta}(x)\}$. We have that if $[\alpha',\, \beta'] \subset [\alpha,\, \beta]$ then $E_{\alpha,\beta} \supset E_{\alpha',\, \beta'}$, and $E_{\alpha,\beta}$ is closed and nowhere dense. Thus

$$E = \bigcup_{\substack{\alpha,\beta\in I\\ \alpha\neq\beta}} E_{\alpha,\beta} = \bigcup_{\substack{\alpha,\beta\in I\cap\mathbb{Q}\\ \alpha\neq\beta}} E_{\alpha,\beta}\,.$$

So E is closed and nowhere dense. Moreover, if $x \in U \setminus E$ and $\alpha < \beta$, we have $\boldsymbol{k_\alpha}(x) < \boldsymbol{k_\beta}(x)$.

(c) The proof is completely similar to that of (d) given below.

(d) Let

$$K = \{x : \lim_{\beta\to b-} \boldsymbol{k_\beta}(x) \not\to +\infty\}$$

If K is not a boundary set then there exist c, d, $c < d$, such that $[c, d] \subset K$. Let

$$M := \sup_{x\in[c,d]} \lim_{\beta\to b-} \boldsymbol{k_\beta}(x)\,.$$

Such a quantity M is clearly finite. We have $\mathfrak{M}_{k_\beta}(\underline{v},\underline{q}) \leq \mathfrak{M}_{e^{Mx}}(\underline{v},\underline{q}) < \max\underline{v}$ for all β and $\underline{v},\underline{q}$ such that $c \leq \min\underline{v} \leq \max\underline{v} \leq d$. Hence the family $\{k_\beta\}$ does not generate a scale on U. So K is dense.

To prove part (B) one needs to show that, under (e), $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is a scale U. By Proposition 1 we know that F is 1–1. Additionally, when arguing to this side, we know that if $x \nearrow x_0$ then $\boldsymbol{k_x} \nearrow \boldsymbol{k_{x_0}}$. So $\boldsymbol{k_x} \rightrightarrows \boldsymbol{k_{x_0}}$ on $[\min\underline{a},\, \max\underline{a}]$. Therefore, by Theorem 3, we have $\mathfrak{M}_{k_x} \rightrightarrows \mathfrak{M}_{k_{x_0}}$ with respect to $\underline{a}$ and $\underline{w}$. Thus F is continuous and 1–1.

To complete the proof, it is sufficient to show that

$$\lim_{\alpha\to a+} F(\alpha) = \min\underline{a}\,, \qquad \lim_{\beta\to b-} F(\beta) = \max\underline{a}\,.$$

We know that $\boldsymbol{k_\beta} \to +\infty$ on the closed interval U. So $\boldsymbol{k_\beta} \rightrightarrows +\infty$ on U. Therefore, for any $M \in \mathbb{R}$ there exists β_M such that

$$F(\beta) \geq \mathfrak{M}_{e^M}(\underline{a},\underline{w})$$

for all $\beta > \beta_M$. Now, taking $M \to +\infty$, and knowing that $\{e^{tx}\colon t \neq 0\} \cup \{x\}$ generates a scale on $\mathbb{R}$ (a folk-type theorem proved in [7]; see also Remark 2) we get

$$F(\beta) \xrightarrow[\beta\to b-]{} \max\underline{a}\,.$$

Similarly one may prove that

$$F(\alpha) \xrightarrow[\alpha\to a+]{} \min\underline{a}\,.$$

So F is a continuous bijection between I and $(\min\underline{a},\, \max\underline{a})$. Hence $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is a scale on U. □

Remark 2. To prove that the family $\{e^{tx}\colon t \neq 0\} \cup \{x\}$ generates a scale on $\mathbb{R}$ it is enough, having data $\underline{a}$, $\underline{w}$, to consider the all-positive-components-vector $\underline{v} = (e^{a_1}, \ldots, e^{a_n})$. And then use the fact that the family of power means evaluated on $\underline{v}$ with weights $\underline{w}$ is a scale on $\mathbb{R}_+$.

Corollary 2 (strenghtening of Proposition 2)**.** *Let U be an interval, $I = (a, b)$ – an open interval, $(k_\alpha)_{\alpha\in I}$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all α.*
(A) *If $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is an increasing scale then there exists an open dense set $X \subset U$ such that*

(a) *if* $\alpha_i \to \alpha+$, *then* $A(k_{\alpha_i}) \to A(k_\alpha)$ *on* X,

(b) *if* $\alpha < \beta$, *then* $A(k_\alpha) < A(k_\beta)$ *on* X,

(c) *if* $\alpha \to a+$, *then* $A(k_\alpha)(x) \to -\infty$ *on* X,

(d) *if* $\beta \to b-$, *then* $A(k_\beta)(x) \to +\infty$ *on* X.

(B) *Under the stronger condition*

(e) *(*$\alpha \to a+ \Rightarrow A(k_\alpha)(x) \to -\infty$*) and (*$\beta \to b- \Rightarrow A(k_\beta)(x) \to +\infty$*) for all* $x \in U$

the entire implication of the corollary can be reversed: (a), (b) *and* (e) *implies that* $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ *is an increasing scale.*

This corollary says that in Proposition 2 one can have a single common subset (X) of U on which conditions (a) through (d) hold.

Proof. We might assume that U is a closed interval (compare the comment in the third paragraph below Proposition 1).

Let $E_{p,q} := \{x \in U\colon\ \boldsymbol{k_p}(x) = \boldsymbol{k_q}(x)\}$. Each $E_{p,q}$ is closed and nowhere dense, so

$$E := \{x\colon \exists_{p,q\in I}\ p \neq q \wedge \boldsymbol{k_p}(x) = \boldsymbol{k_q}(x)\}$$

has two more descriptions

$$E = \bigcup_{\substack{\alpha,\beta\in I\\ \alpha\neq\beta}} E_{\alpha,\beta} = \bigcup_{\substack{\alpha,\beta\in I\cap\mathbb{Q}\\ \alpha\neq\beta}} E_{\alpha,\beta}\,.$$

We know that E is closed nowhere dense, being a countable union of closed nowhere dense sets. So $X_{\neq} := U \setminus E$ is an open dense set. Let

$$X_{+\infty} := \{x \in U\colon\ \lim_{\beta\to b-} \boldsymbol{k_\beta}(x) \to +\infty\}.$$

By Proposition 2 we know that $X_{+\infty}$ is dense. We now prove that it is open. Let

$$X_s := \{x \in U\colon\ \lim_{\beta\to b-} \boldsymbol{k_\beta}(x) > s\}.$$

Observe that X_s is dense (because $X_s \supset X_{+\infty}$). Moreover, for all $x_0 \in X_s$ there holds $\boldsymbol{k_{\beta_0}}(x_0) > s + \delta$ for some $\beta_0 \in I$ and $\delta > 0$. Hence one may take an open neighborhood $P \ni x_0$ satisfying $\boldsymbol{k_{\beta_0}}(x) > s + \frac{1}{2}\delta$ for all $x \in P$, implying $P \subset X_s$. So X_s is open. But the mapping $\beta \mapsto \boldsymbol{k_\beta}(x)$ is nondecreasing for all $x \in U$. Hence $X_{+\infty} = \bigcap_{n=1}^{\infty} X_n$ is open and dense. Similarly,

$$X_{-\infty} := \{x \in U\colon\ \lim_{\alpha\to a+} \boldsymbol{k_\alpha}(x) \to -\infty\}$$

is an open dense set as well. Now one may take $X := X_{\neq} \cap X_{-\infty} \cap X_{+\infty}$.
X is clearly open and dense. □

4 Proofs of Theorems 1 and 2

Proof of Theorem 1. Let U be an interval, $I = (a, b)$ an open interval, X be a dense subset of U, where the mapping given in the wording of theorem is increasing and 1–1. We work with the family of functions $(k_\alpha)_{\alpha\in I}$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all α.

Let us take an arbitrary $x_0 \in X$. We know that $I \ni \alpha \mapsto \boldsymbol{k_\alpha}(x_0)$ is increasing, 1–1 and onto $\mathbb{R}$. Next, let us specify the function $\Phi : \mathbb{R} \to I$ such that $\boldsymbol{k_{\Phi(\alpha)}}(x_0) = \alpha$. This function is increasing as well.

Then for $\alpha < \beta$ we have $\boldsymbol{k}_{\boldsymbol{\Phi(\alpha)}} < \boldsymbol{k}_{\boldsymbol{\Phi(\beta)}}$ on the dense subset of U emerging from Corollary 2. Due to the fact that $I \ni \alpha \mapsto \boldsymbol{k}_{\boldsymbol{\alpha}}(x) \in \mathbb{R}$ is onto, we have

$$\lim_{\alpha \to a} \boldsymbol{k}_{\boldsymbol{\Phi(\alpha)}}(x) = -\infty \qquad \text{and} \qquad \lim_{\beta \to b} \boldsymbol{k}_{\boldsymbol{\Phi(\beta)}}(x) = +\infty$$

everywhere on U. So, using the part (B) of Corollary 2, the family of means $(\mathfrak{M}_{k_\alpha})_{\alpha \in I}$ is an increasing scale on U. □

Proof of Theorem 2. Let us take X from Corollary 2. Let then fix any $x_0 \in X$. Let $\{s_p\}_{p \in \mathbb{R}}$ be the reparametrized family $\{k_\alpha\}_{\alpha \in I}$, with restriction

$$s_p = k_\alpha \text{ , where } p = \boldsymbol{k}_{\boldsymbol{\alpha}}(x_0).$$

Then we know that the mapping

$$\mathbb{R} \ni p \mapsto \boldsymbol{s}_{\boldsymbol{p}}(x) \in \mathbb{R}$$

is 1–1 and onto for all $x \in X$, and if $p > q$

$$\boldsymbol{s}_{\boldsymbol{p}}(x) > \boldsymbol{s}_{\boldsymbol{q}}(x).$$

Moreover, due to the fact that $\boldsymbol{s}_{\boldsymbol{p}}(x_0)$ is onto, we have for all x_0

$$\lim_{p \to -\infty} \boldsymbol{s}_{\boldsymbol{p}}(x_0) = -\infty \qquad \text{and} \qquad \lim_{p \to +\infty} \boldsymbol{s}_{\boldsymbol{p}}(x_0) = +\infty\,.$$

So $p \mapsto \boldsymbol{s}_{\boldsymbol{p}}(x)$ is increasing, 1–1 and onto $\mathbb{R}$ for all $x \in X$. □

5 Applications

Proposition 3 (power means do generate a scale)**.** *Let $U = \mathbb{R}_+$ and $(r_\alpha)_{\alpha \in \mathbb{R}}$, given by*

$$r_\alpha(x) = \begin{cases} x^\alpha & \alpha \neq 0 \\ \ln x & \alpha = 0 \end{cases},$$

be the family of power functions. Then the family (r_α) generates a scale on $\mathbb{R}_+$.

Proof. We compute $\boldsymbol{r}_{\boldsymbol{\alpha}}$,

$$\boldsymbol{r}_{\boldsymbol{\alpha}}(x) \;=\; \frac{\alpha - 1}{x}$$

and see that the mapping $\alpha \mapsto \boldsymbol{r}_{\boldsymbol{\alpha}}(x)$ is increasing, 1–1 and onto for all $x \in \mathbb{R}_+$. So the assumptions in Theorem 1 hold, implying that the family (r_α) generates an increasing scale on $\mathbb{R}_+$. □

Before giving our second application we reproduce a 10 years old' result.

Proposition 4 ([K])**.** *Let $g\colon [0,1] \to \mathbb{R}$ be a continuous monotone function. Writing $g_\alpha(x) := g(x^\alpha)$ for any $\alpha > 0$, there hold:*

(i) *if there exists the one side, nonzero derivative $g'(0+)$ then*

$$\lim_{\alpha \to +\infty} \mathfrak{M}_{g_\alpha} \;=\; \max\,,$$

(ii) *if there exists one side, nonzero derivative $g'(1-)$ then*

$$\lim_{\alpha \to 0+} \mathfrak{M}_{g_\alpha} \;=\; \mathfrak{M}_{\ln x}\,.$$

We prove a somehow similar, yet not so close, result.

Proposition 5. *Let $g \in \mathcal{C}^{2\neq}[0,1] \to (0, +\infty)$ (i.e., there exist the relevant one side second derivatives of g at 0 and 1, too) and $g_\alpha(x) := g(x^\alpha)$, $\alpha \in (0, +\infty)$. Then*

$$\lim_{\alpha\to 0+} \mathfrak{M}_{g_\alpha} = \mathfrak{M}_{\ln x} \qquad \textit{and} \qquad \lim_{\alpha\to +\infty} \mathfrak{M}_{g_\alpha} = \max . \tag{3}$$

If, in addition, g is convex,[3] *then $(g_\alpha)_{\alpha\in(0,+\infty)}$ generates a scale between the geometric mean and* max.

Proof. We have to prove that the mapping $(0, +\infty) \ni \alpha \mapsto \boldsymbol{g_\alpha}(x) \in \mathbb{R}$ is 1–1 and onto for all $x \in (0, 1)$. Let us fix an arbitrary $x \in (0, 1)$. Then we have

$$\boldsymbol{g_\alpha}(x) \;=\; \alpha x^{\alpha-1}\boldsymbol{g}(x^\alpha) + \frac{\alpha-1}{x} .$$

When $\alpha \to 0+$, then

$$\boldsymbol{g_0}(x) := \lim_{\alpha\to 0+} \boldsymbol{g_\alpha}(x) \;=\; \frac{-1}{x} .$$

In turn, when $\alpha \to +\infty$, there holds

$$\boldsymbol{g_\alpha}(x) \;=\; \underbrace{\alpha x^{\alpha-1}\frac{g''(0)}{g'(0)}}_{>-\infty} + \frac{\alpha-1}{x} \to +\infty .$$

The proof of formulas (3) is now completed.

When, additionally, g is convex, then $\boldsymbol{g} \geq 0$ and, by Corollary 1, the family $\{g_\alpha\}_{\alpha\in\mathbb{R}_+}$ generates a scale on $(0,1)$ between the geometric mean and max. $\square$

Now we would like to present one classical result of the Italian School of statisticians from 1910-20s. That result has been reported in [B, p. 269]. We now give it a new short proof based on Corollary 1.

Proposition 6 (Radical Means)**.** *Let $U = \mathbb{R}_+$ and $(k_\alpha)_{\alpha\in\mathbb{R}_+}$, $k_\alpha(x) = \alpha^{1/x}$, be the family of radical functions. Then this family generates a decreasing scale on $\mathbb{R}_+$.*

Proof. The proof appears to be quite close to the proof of Proposition 3. Indeed, we quickly compute

$$\boldsymbol{k_\alpha}(x) = -\frac{2x + \ln\alpha}{x^2} ,$$

finding that the mapping $\alpha \mapsto \boldsymbol{k_\alpha}(x)$ is decreasing, 1–1 and onto for all $x \in \mathbb{R}_+$. So the assumption in Corollary 1 hold, and hence the family $(k_\alpha)_{\alpha\in\mathbb{R}_+}$ generates a decreasing scale on $\mathbb{R}_+$. $\square$

Open problem. How to unify Theorem 1 and Theorem 2 so as to get a set of conditions that would simultaneously be necessary and sufficient?

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[3] in this situation we can just assume that $g \in \mathcal{C}^2[0,1)$, instead of assuming $g \in \mathcal{C}^{2\neq}[0,1)$, because all convex, strictly monotone functions in $\mathcal{C}^2[0,1)$ belong to $\mathcal{C}^{2\neq}[0,1)$.

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